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ELASTIC–PLASTIC WAVE GENERATION BY IMPULSIVE ELECTROMAGNETIC RADIATION*

L. W. MORLAND†

Department of the Aerospace and Mechanical Engineering Sciences, University of California, San Diego, La Jolla, California

Abstract—Absorption of radiation through a thin layer adjacent to an irradiated, stress-free, surface causes rapid heating and the generation of stress waves. Temperature rises sufficient to cause plastic yielding are considered, and a resulting elastic–plastic wave solution is constructed. Comparison is made with the purely elastic solution, and in particular the reduced amplitude and pulse spread of the outgoing tensile wave is illustrated.

1. INTRODUCTION

THE penetration of electromagnetic radiation into a thin surface layer of a solid, and the consequent, effectively instantaneous, absorption of heat, causes non-uniform temperature rise through the layer, which in turn generates stress waves. The stress levels attained depend on the temperature rise which is governed by the total energy deposited and the thickness of the absorption layer (absorption depth). In particular, if the radiation pulse is of short duration compared with stress wave travel time through the absorption layer, itself much shorter than the thermal diffusion time, there is an essentially instantaneous non-uniform rise of temperature in the layer, which then remains effectively steady during the subsequent wave propagation over several absorption depths. The impulsive limit was exploited by Morland [1] in a linear thermoelastic analysis to demonstrate the simple wave pattern generated, and in particular the significant tensile wave which is formed within two absorption depths when the irradiated surface is stress-free. Analogous solutions for thermalviscoelastic and thermoelastic materials with temperature dependent properties, for which the steady temperature field was an essential simplifying feature, were subsequently obtained by Hegemier and Morland [2] and Hegemier and Tzung [3] respectively. An elastic–plastic analysis is now presented.

Time scales considered in the elastic and viscoelastic treatments were based on a radiation duration of order 10^{-8} sec appropriate to a Q-switched laser, and a wave travel time through the layer greater than 10^{-6} sec assuming a typical wave speed of order 10^5 cm/sec and absorption depth greater than 10^{-1} cm. Regarding the 10^{-8} sec pulse duration as a lower bound, the impulsive limit is applicable only for absorption depths greater than 10^{-1} cm. While many elastic and viscoelastic solids are transparent to optical frequencies, other frequency bands will be required for such absorption in metals. Furthermore, to achieve the higher temperature rise necessary to generate stress levels beyond the plastic yield limit, the energy deposit per unit area of irradiated surface must be significantly greater than the 2 cal/cm^2 used for the temperature estimates in [1, 2]. Surface

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† Consultant. Permanent Position: School of Mathematics and Physics, University of East Anglia, England.

temperature rises of 130°–340°F will suffice to cause yield in a number of metals on the basis of the room temperature yield stress, and less if the yield stress is lowered at higher temperatures. A contrasting effect would be a rise in yield stress at high stress rates. This method of wave generation may prove a useful technique for testing such thermal and dynamic properties. In particular the effective yield stress at the surface temperature is a distinct parameter of the wave solution.

If the surface temperature rise is higher than the minimum level for yield by a factor 2 (an estimate based on equal tensile and compressive yield stress) then reverse yield (opposite shear sense) may also occur and a much more complicated wave pattern will result. The present analysis is restricted to the intermediate temperature rise with yield in one sense only. A formal solution is constructed for a temperature distribution decreasing away from the surface and a yield stress which is a decreasing (or constant) function of temperature. Uniaxial displacement is assumed as the appropriate approximation for short propagation distances (say 2–5 absorption depths) compared with the dimensions of the irradiated area. An elastic–plastic wave pattern is motivated by the elastic solution [1], and validity in the different regions is either proved or strongly inferred under certain restrictions on the prescribed functions. It is confirmed in several numerical illustrations.

In the elastic solution [1] there is an initial build-up of compressive stress with maximum value (adopted as the stress unit) adjacent to the free surface, so that the unit discontinuity (in tensile sense) propagates away from the surface. Within a few absorption depths the final behaviour is effectively attained, in which the discontinuity raises the stress from a compressive level $-0{\cdot}5$ to a tensile level $0{\cdot}5$. In the present solution the discontinuity is propagated in two parts, an elastic forerunner with magnitude depending on the yield stress followed by a slower plastic discontinuity, with the separating region containing continuous elastic waves. The plastic discontinuity attenuates, and in the numerical examples is annulled before two absorption depths. Maximum tensile stresses behind the plastic discontinuity are less than 0·5, and in the limit case appropriate to this intermediate temperature range the maximum is less than 0·38. There is a slight increase in the continuing (continuous) elastic wave beyond the attenuation point. Apart from the reduction in the maximum tensile stress, there is a spread of the tensile loading part of the final outgoing wave, discontinuous in the elastic solution.

2. GOVERNING EQUATIONS

Let x be a material coordinate denoting particle distance from the surface in the undeformed configuration, and t denote time following the radiation impulse. In the region behind the irradiated area, for the distances and times considered, it is assumed that the temperature T, initially zero, varies only in the x-direction, and that there is a uniaxial particle displacement in the x-direction, $u(x, t)$.

If the impulsive radiation effects an instantaneous heat deposit per unit volume $q(x)$, the corresponding temperature rise $T(x)$ is given by

$$q(x) = \rho \int_0^{T(x)} C_v(t)\, \mathrm{d}T, \qquad T(x) = \frac{q(x)}{\rho \bar{C}_v}, \tag{2.1}$$

where ρ is the density in the undeformed state, $C_v(T)$ is the specific heat at constant density; and $\bar{C}_v$ denotes the appropriate mean value.* It is assumed that $q'(x) < 0$; $q(x), q'(x) \to 0$ as $x \to \infty$, and the main decrease occurs within an absorption depth l defined by

$$\int_0^\infty q(x)\,\mathrm{d}x = Q = q(0)l, \tag{2.2}$$

where Q is the total energy deposit per unit area of surface. Thus

$$T(x) = T_M f(x), \qquad T_M = \frac{Q}{\rho l \bar{C}_v}, \qquad f(x) = \frac{q(x)}{q(0)}, \qquad f'(x) < 0, \tag{2.3}$$

and T_M is the surface (maximum) temperature rise. Since heat conduction through the material (and across the surface) will be negligible during the time for wave travel over a few absorption depths, the temperature remains effectively at the steady distribution (2.3).

With the restriction to uniaxial displacement there is a principal longitudinal (Cauchy) stress component σ, and equal principal lateral components σ_L in each normal direction to the x-axis. The overall strain has a single principal longitudinal component, ε, but during plastic deformation there are non-zero elastic (recoverable) and plastic (permanent) parts of a zero total principal strain ε_L. In incremental form

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon^e + \mathrm{d}\varepsilon^p, \qquad \mathrm{d}\varepsilon_L = \mathrm{d}\varepsilon_L^e + \mathrm{d}\varepsilon_L^p = 0, \tag{2.4}$$

where the superscripts e, p denote elastic and plastic parts respectively, and an increment $\mathrm{d}\varepsilon$ defines the classical strain measure: extension per unit current length. The elastic strain increments are governed by the Hooke's laws with incorporated thermal expansion contributions, here reducing to

$$\begin{aligned} 3GK\,\mathrm{d}\varepsilon^e &= (K+\tfrac{1}{3}G)\,\mathrm{d}\sigma - (K-\tfrac{2}{3}G)\,\mathrm{d}\sigma_L + 3GK\alpha\,\mathrm{d}T \\ 6GK\,\mathrm{d}\varepsilon_L^e &= (K+\tfrac{4}{3}G)\,\mathrm{d}\sigma_L - (K-\tfrac{2}{3}G)\,\mathrm{d}\sigma + 6GK\alpha\,\mathrm{d}T, \end{aligned} \tag{2.5}$$

where the bulk and shear moduli K, G, and coefficient of linear expansion α, may be functions of the current stress and temperature. In particular, when plastic yielding is not taking place, $\mathrm{d}\varepsilon_L^e \equiv 0$ and the pure elastic changes satisfy

$$\begin{aligned} (K+\tfrac{4}{3}G)\,\mathrm{d}\varepsilon &= \mathrm{d}\sigma + 3K\alpha\,\mathrm{d}T, \\ (K+\tfrac{4}{3}G)\,\mathrm{d}\sigma_L &= (K-\tfrac{2}{3}G)\,\mathrm{d}\sigma - 6GK\alpha\,\mathrm{d}T. \end{aligned} \tag{2.6}$$

Both the von Mises and Tresca yield criteria reduce to

$$\sigma - \sigma_L = \pm y, \tag{2.7}$$

where y is the yield stress in simple tension, in general a function of the work-hardening, and possibly of temperature. While quasistatic tests for many metals indicate significant temperature dependence after long exposure to the new temperature [4], a corresponding change of property within wave travel times of order 10^{-6} sec is not established, but solutions will be constructed for both temperature dependent and independent yield stress. It will be assumed, as in quasistatic data, that the yield stress decreases with temperature. The typical work-hardening dependence makes little contribution in this stress

* In [1] this result was deduced by a limit process.

geometry, except perhaps near initial yield, as indicated in a fuller discussion of the equations, without thermal terms, by Morland [5], and will be neglected. Strain-rate effects too are ignored, with the exception of possible adjustment to the value of $y(T)$ to describe effects over the band of high rates occurring here. Thus

$$y = y(T), \qquad y'(T) \leq 0. \tag{2.8}$$

The yield behaviour is fully specified by adjoining the postulate of plastic incompressibility,

$$\mathrm{d}\varepsilon^p + 2\,\mathrm{d}\varepsilon_L^p = 0. \tag{2.9}$$

Eliminating $\mathrm{d}\varepsilon_L^p$ between (2.4), (2.9), $\mathrm{d}\varepsilon_L^e$ by (2.5), and $\mathrm{d}\sigma_L$ by (2.7) gives the longitudinal plastic relation

$$K\,\mathrm{d}\varepsilon = \mathrm{d}\sigma + [3K\alpha \mp \tfrac{2}{3}y']\,\mathrm{d}T. \tag{2.10}$$

For a valid plastic deformation the plastic work done must be non-negative, that is

$$\mathrm{d}W_p = \sigma\,\mathrm{d}\varepsilon^p + 2\sigma_L\,\mathrm{d}\varepsilon_L^p \geq 0. \tag{2.11}$$

Eliminating $\mathrm{d}\varepsilon^p$ by (2.9), $\mathrm{d}\varepsilon_L^p$ by (2.4), and using (2.7), (2.5),

$$3GK\,\mathrm{d}W_p = \pm y[2G\,\mathrm{d}\sigma + \{6GK\alpha \mp (K + \tfrac{4}{3}G)y'\}\,\mathrm{d}T]. \tag{2.12}$$

In particular, at a fixed particle with steady temperature $T(x)$, $\mathrm{d}T = 0$ and the validity requirement becomes

$$\sigma - \sigma_L = \pm y \to \mathrm{d}\sigma \gtrless 0. \tag{2.13}$$

For convenience, describe the positive sign condition as yield, and the negative sign condition as reverse yield. Finally, elastic validity requires

$$|\sigma - \sigma_L| < y, \quad \text{or} \quad \sigma - \sigma_L = \pm y \quad \text{and} \quad \mathrm{d}(\sigma - \sigma_L) \lesseqgtr 0. \tag{2.14}$$

The subsequent wave analysis is considerably simplified by assuming that the moduli K, G are independent of the stress, and further by equating the strain to the engineering measure, extension per unit initial length,

$$\varepsilon = \frac{\partial u}{\partial x}. \tag{2.15}$$

Such neglect of curvature in the resulting stress–strain laws has been investigated in [5] for stress waves with amplitude of order five times the yield stress in an aluminium alloy, and was found not to be significant in the overall wave patterns and amplitudes. This simplification is adopted here.

In the initial (instantaneous) heat deposit the strain remains zero and there is a consequent build-up of isotropic pressure given by (2.6):

$$\sigma = \sigma_L = -3\int_0^{T(x)} K(T)\alpha(T)\,\mathrm{d}T, \tag{2.16}$$

which is purely elastic. During the subsequent deformation, with $T = T(x)$, the stress–strain laws for elastic and plastic changes are given by (2.6), (2.10) respectively with $\mathrm{d}T = 0$.

Both are linear, but with slopes depending on x when K, G vary with T, resulting in non-homogeneous wave equations for the stress. An elastic analysis is presented in [3] assuming a large temperature range and moduli variation by a factor two, and differences from the homogeneous case are significant but not large. For more modest temperature dependence it is unlikely that this effect would be important, and will be neglected here. Elastic and plastic stress–strain paths then have respectively the same constant slopes $K+\frac{4}{3}G$ and K.

A longitudinal $\sigma-\varepsilon$ path applicable to a typical particle $x(>0)$ is shown in Fig. 1.

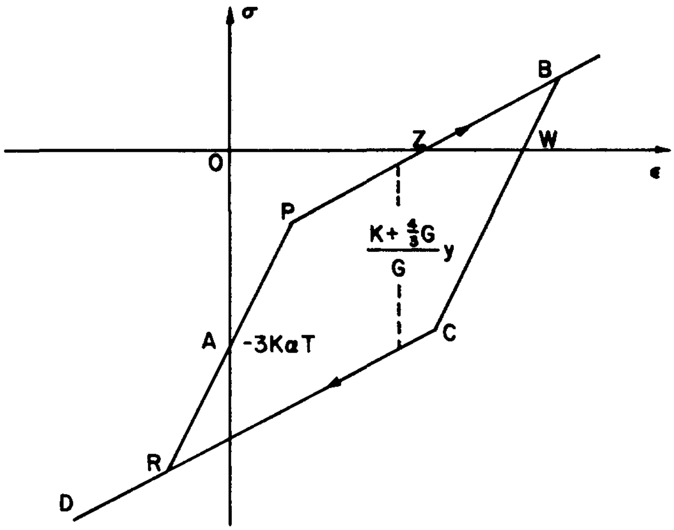

FIG. 1. Longitudinal stress–strain path at steady temperature T.

The initial stress jump at zero strain to a compressive stress $-3K\alpha T(x)$ is represented by OA, and subsequent steady temperature elastic changes, prior to yield, are restricted to the path RP. Along RP, by (2.6)

$$(K+\tfrac{4}{3}G)\,\mathrm{d}(\sigma-\sigma_L) = 2G\,\mathrm{d}\sigma, \tag{2.17}$$

and the initial yield points P, R are given by

$$\sigma = -3K\alpha T \pm \frac{K+\frac{4}{3}G}{2G}y. \tag{2.18}$$

Subsequent yield or reverse yield follows the irreversible plastic paths PB, RD respectively, with the one-way direction indicated by the arrows. Decrease of σ from a plastic state B follows the reversible elastic path BC until the new reverse yield point C is reached after a stress drop $(K+\frac{4}{3}G)y/G$, and then follows the reverse yield path CD. For different particles the initial stress $-3K\alpha T(x)$ is different, and if $y = y(T)$ the elastic stress-range $(K+\frac{4}{3}G)y/G$ varies. The requirements on $T(x)$ for yield to occur during the wave propagation are given in the next section.

Finally, the momentum equation

$$\frac{\partial\sigma}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{2.19}$$

on differentiating with respect to x, using the strain definition (2.15) and elastic and plastic laws (2.6), (2.10) respectively, reduces to

$$\frac{\partial^2 \sigma}{\partial x^2} = \frac{1}{c_i^2}\frac{\partial^2 \sigma}{\partial t^2}, \qquad (i = 1, 2) \tag{2.20}$$

where c_0 and c_1 are respectively the elastic and plastic wave speeds given by

$$\rho c_0^2 = K + \tfrac{4}{3}G, \qquad \rho c_1^2 = K (< \rho c_0^2). \tag{2.21}$$

Note that these wave equations are obtained also for T linear in t provided that K, G, y are independent of T, and, as remarked earlier, for temperature dependent K, G, y with $T = T(x)$ when the wave speeds depend on T and hence on x. To determine the propagation behaviour of a stress discontinuity we must further postulate the instantaneously followed stress–strain path. A fuller discussion is given by Morland and Cox [6], leading to the adoption of the corresponding path for a continuous change, from which it follows that elastic jumps propagate with speed c_0 and plastic jumps with speed c_1. In particular a jump embracing both elastic and plastic changes splits into the two parts propagating with their respective speeds.

3. PLASTICITY INFLUENCE ON WAVE PATTERN

First introduce dimensionless distance and time coordinates ξ, η by

$$x = l\xi, \qquad t = l\eta/c_0, \tag{3.1}$$

and define dimensionless stress $\theta(\xi, \eta)$, and effective yield stress $Y(\xi)$, with the initial maximum compressive stress as unit; thus

$$\sigma_M = 3K\alpha T_M, \qquad \sigma = \sigma_M \theta, \qquad (K + \tfrac{4}{3}G)y = 2G\sigma_M Y, \qquad Y'(\xi) \geq 0. \tag{3.2}$$

The initial stress distribution is given by (2.16), (2.3),

$$\theta(\xi, 0) = -f(x) = E(\xi), \qquad \xi > 0. \tag{3.3}$$

where

$$E(0) = -1, \qquad E(\xi) \leq 0, \qquad E'(\xi) > 0; \qquad E(\xi), E'(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{3.4}$$

If we further postulate that $f''(x) \geq 0$, and note the common observation [4] $y''(T) \leq 0$, then it follows that

$$E''(\xi) \leq 0, \qquad Y''(\xi) \leq 0, \tag{3.5}$$

which are useful conditions for an analytic discussion of the wave solution validity. The second initial condition is zero stress rate, derived in [1], and a stress-free boundary condition is assumed:

$$\frac{\partial \theta}{\partial \eta}(\xi, 0) = 0, \qquad \xi > 0; \qquad \theta(0, \eta) = 0, \qquad \eta > 0. \tag{3.6}$$

If yield does not occur, then the solution everywhere satisfies the elastic wave equation subject to the initial and boundary conditions (3.3), (3.6), and is simply

$$\theta = \begin{cases} \tfrac{1}{2}E(\xi+\eta)+\tfrac{1}{2}E(\xi-\eta), & \eta < \xi \\ \tfrac{1}{2}E(\xi+\eta)-\tfrac{1}{2}E(\eta-\xi), & \eta > \xi. \end{cases} \tag{3.7}$$

The solution in [1] was presented only for $E(\xi) = -e^{-\xi}$. There is a constant stress jump $-E(0) = 1$ across the characteristic $\eta = \xi$, OM in Fig. 2. The assumption $E''(\xi) \leq 0$ is

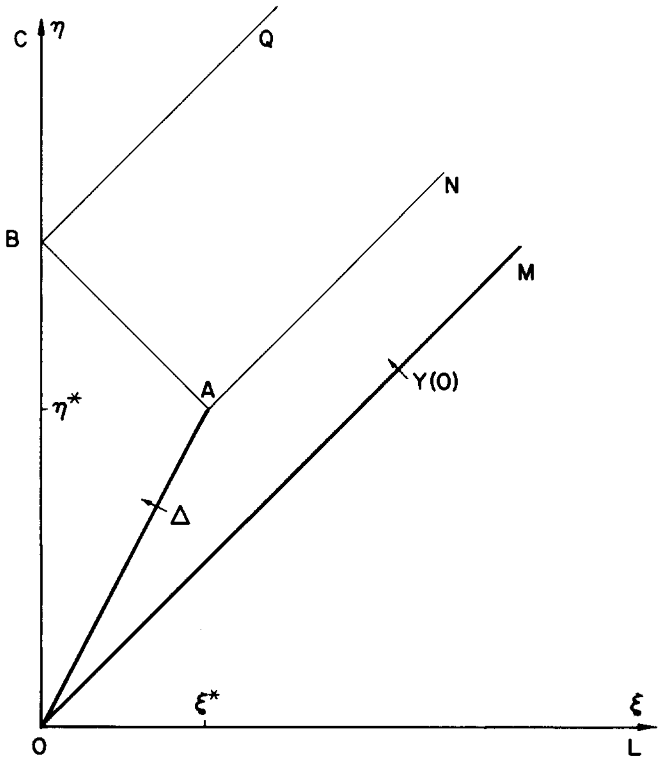

FIG. 2. Different solution domains in the ξ, η plane: stress discontinuity paths shown ———.

sufficient to guarantee that the stress at fixed ξ decreases from $\eta = 0$ to $\eta = \xi$ (increasing compressive stress), and following the jump to a tensile stress at $\eta = \xi$, decreases to zero as $\eta \to \infty$. Thus the maximum compressive and tensile stress at each ξ occur on $\eta = \xi-$ and $\xi+$ respectively, and further, these maxima decrease and increase respectively as $\xi \to \infty$. The maximum compressive stress is -1, at $\xi = 0$, and the maximum tensile stress is 0·5, approached in the limit as $\xi \to \infty$.

Introducing a dimensionless yield function $D(\xi, \eta)$, the initial yield criterion is given by

$$D(\xi, \eta) = \theta(\xi, \eta) - E(\xi), = \pm Y(\xi), \tag{3.8}$$

corresponding to the points P, R in Fig. 1. In the elastic solution (3.7), at each fixed ξ and η increasing, D decreases from 0 at $\eta = 0$ to a negative value $-[\tfrac{1}{2}-\tfrac{1}{2}E(2\xi)+E(\xi)]$ on $\eta = \xi-$, jumps to a positive value $[\frac{1}{2}-E(\xi)+\frac{1}{2}E(2\xi)]$ on $\eta = \xi+$, then decreases to $-E(\xi)$, >0 as $\eta \to \infty$. Further, $D(\xi, \xi+)$ decreases with increase of ξ, in view of (3.5), so $D(\xi, \eta)$ has a maximum value $D(0, 0+) = 1$. Thus yield in the positive sense would first occur at $(0, 0)$ if $Y(0) \leq 1$, recalling that $Y'(\xi) \geq 0$. Similarly, $-D(\xi, \xi-)$ increases with ξ, approaching a limit value 0·5 as $\xi \to \infty$, so that reverse yield will not occur if $Y(0) \geq 0{\cdot}5$. With a varying yield stress such that $Y(0) < 0{\cdot}5$, $Y(\infty) \geq 0{\cdot}5$, reverse yield may, or may not, occur, but certainly occurs if $Y(\infty) < 0{\cdot}5$. Under a further postulate $E'''(\xi) > 0$, satisfied by $-e^{-\xi}$, it also follows in $\eta < \xi$ that $-D(\xi, \eta)$ attains its maximum value for fixed η on $\eta = \xi-$ so that reverse yield would first occur at (ξ_R, ξ_R-), where

$$1-E(2\xi_R)+2E(\xi_R) = 2Y(\xi_R), \tag{3.9}$$

provided that (3.9) has a solution. Attention is now restricted to the intermediate temperature range which produces a maximum stress σ_M such that

$$0{\cdot}5 \leq Y(0) < 1, \tag{3.10}$$

and reverse yield does not occur in the domain LOM, Fig. 2.

Before proceeding to the wave analysis, estimates based on quasistatic, room temperature, data for various metals can be made to determine the temperature rise T_M required to initiate yield; that is, to produce a value σ_M such that $Y = 1$, recalling the definitions (3.2). If a lower yield stress applies at the higher surface temperature, at these high rates of strain, then the required rise T_M will be lower—and vice versa. Values of y, K, G, α for a carbon steel, a magnesium alloy, manganese bronze, and an aluminum alloy, [4], give approximate temperature rises 130, 140, 160 and 340°F respectively. A distinct case is titanium which requires 1200°F. For the main group, a ratio of plastic and elastic wave speeds c_1/c_0, defined in (2.21), has the same approximate value 0·8.

For $Y(0) < 1$, at $\xi = 0+$ and time $\eta = 0$ the stress jumps from -1 to $-1+Y(0)$ elastically, the path AP in Fig. 1, and the remaining jump to reach the zero boundary stress takes place plastically, the path PZ. The most simple wave pattern consistent with this modified behaviour at $(0, 0)$ assumes the propagation of a reduced elastic jump of magnitude $Y(0)$ along the characteristic OM, Fig. 2, and an attenuating plastic jump $\Delta(\xi)$ along the plastic characteristic OA, being annulled at A, $\xi = \xi^*$. That is

$$\Delta(0) = 1-Y(0) > 0, \qquad \Delta(\xi^*) = 0. \tag{3.11}$$

Since a plastic wave continuously dissipates energy, and after the initial finite deposit there is no further input, the wave must attenuate, and without loss of generality a finite annullment distance ξ^* may be designated in the assumed wave pattern. It is further assumed that yield does not occur in the domain $MOAN$, and clearly the elastic solution in LOM is not influenced by the above modification.

Thus as η increases from zero, the stress at a typical particle ξ in the range $0 < \xi < \xi^*$ decreases monotonically from $E(\xi)$ until $\eta = \xi$, along the elastic path AR, but not passing R, in Fig. 1; undergoes an elastic jump $Y(0)$ at $\eta = \xi$ along RP, but not passing P; then makes continuous elastic changes, either sense, within RP until the plastic characteristic OA, Fig. 2, is reached. At OA, by assumption, the stress is at the local yield value $Y(\xi)$, the point P in Fig. 1, and across OA makes a jump $\Delta(\xi)$ following a plastic path PB. The condition $\Delta > 0$ ensures that the change takes place in the allowed direction PB. Motivated by the elastic solution it is anticipated that tensile stresses are attained behind the plastic jump, so that B is beyond Z, and that in the subsequent continuous change back to zero stress, the stress never exceeds the level at B (nor drops below C); that is, remains on the elastic path BC.

At the annullment point A, Fig. 2, the stress–strain state is the point P, Fig. 1, and for $\xi > \xi^*$ all changes take place along the elastic path RP. These wave pattern assumptions must be confirmed after the formal solution is constructed.

4. WAVE SOLUTION

Defining the ratio of plastic and elastic wave speeds by

$$c = c_1/c_0, < 1, \tag{4.1}$$

the plastic characteristic OA, Fig. 2, is described by points

$$(\xi, \xi/c), \qquad 0 \le \xi \le \xi^*. \tag{4.2}$$

The validity requirements imposed by the wave pattern assumptions are

$$\left.\begin{array}{ll} \xi \le \xi^*, & 0 \le \eta < \xi/c; \\ \xi > \xi^*, & \eta \ge 0: \end{array}\right. \qquad |D(\xi, \eta)| < Y(\xi); \tag{4.3}$$

$$\xi \le \xi^*, \qquad \eta > \xi/c: \qquad -2Y(\xi) < \theta(\xi, \eta) - \theta(\xi, \xi/c) < 0; \tag{4.4}$$

$$\xi < \xi^*, \qquad \eta = \xi/c: \qquad \Delta(\xi) > 0. \tag{4.5}$$

The plastic work condition (4.5) is ensured by the definition (3.11) of ξ^*, and the elastic condition (4.3) is satisfied in the domain LOM under the restriction (3.10).

With the exception of the discontinuity paths OM, OA, the stress in all other domains of the ξ, η plane is described by continuous elastic waves, with unit speed $d\xi/d\eta$, moving in the positive and negative $\xi-$ directions. Introducing a dimensionless particle velocity

$$V = \frac{\rho c_0}{\sigma_M} \frac{\partial u}{\partial t}, \tag{4.6}$$

the jump condition across the elastic path OM is

$$[V] = -Y(0), \tag{4.7}$$

and across the plastic path OA,

$$[V] = -\Delta(\xi)/c, \tag{4.8}$$

while in the continuous domains,

$$\frac{\partial V}{\partial \eta} = \frac{\partial \theta}{\partial \xi}. \tag{4.9}$$

The solution is obtained simply by expressing the stress in pairs of wave functions in each of the different influence domains shown in Fig. 2, and satisfying the initial and boundary conditions (3.3), (3.6), together with the jump conditions (4.7), (4.8). The stress

$\theta(\xi, \eta)$ is given as follows:

$$LOM: \quad \tfrac{1}{2}E(\xi+\eta)+\tfrac{1}{2}E(\xi-\eta). \tag{4.10}$$

$$MOAN: \quad \tfrac{1}{2}E(\xi+\eta)+E\left[\frac{c}{1-c}(\eta-\xi)\right]-\tfrac{1}{2}E\left[\frac{1+c}{1-c}(\eta-\xi)\right]+Y\left[\frac{c}{1-c}(\eta-\xi)\right]. \tag{4.11}$$

$$QAB: \quad j(\xi+\eta)-j(\eta-\xi). \tag{4.12}$$

$$NABQ: \quad \tfrac{1}{2}E(\xi+\eta)-j(\eta-\xi)+j\left(\frac{1+c}{c}\xi^*\right)-\tfrac{1}{2}E\left(\frac{1+c}{c}\xi^*\right). \tag{4.13}$$

$$QBC: \quad \tfrac{1}{2}E(\xi+\eta)-\tfrac{1}{2}E(\eta-\xi). \tag{4.14}$$

The single unspecified function $j(z)$ is defined by a difference equation

$$j(z)-\delta j(\delta z) = G(z), \qquad z \geq 0, \tag{4.15}$$

where

$$\delta = \frac{1-c}{1+c} < 1, \qquad z = \frac{1+c}{c}\xi > \xi, \tag{4.16}$$

$$G(z) = \tfrac{1}{2}(1-\delta)[1+E(z)]+\delta[1+E(\xi)]+\delta[Y(\xi)-Y(0)]. \tag{4.17}$$

Finally, the formal solution is completed by determing the (first) zero, ξ^*, of the plastic stress jump

$$\Delta(\xi) = 1-Y(0)+\frac{1-\delta}{2\delta}[1+E(z)]-\frac{1-\delta}{\delta}j(z). \tag{4.18}$$

Since $\delta < 1$, and moreover is typically small recalling that c is commonly of order 0·8, a series solution of (4.15) is obtained by iteration, thus

$$j(z) = \sum_{r=0}^{\infty} \delta^r G(\delta^r z). \tag{4.19}$$

In particular, for small δ and $Y(\xi)$ bounded by $1/\delta$, few terms are needed for high accuracy. However, various properties of the wave solution including most of the validity requirements can be deduced without the explicit solution $j(z)$. First note that using (3.2), (3.4), (3.5),

$$G(0) = 0; \qquad G'(z) > 0, G''(z) \leq 0, \qquad z \geq 0. \tag{4.20}$$

It then follows from (4.15) that

$$j(0) = 0; \qquad j'(z) > 0, j''(z) \leq 0, \qquad z \geq 0, \tag{4.21}$$

where the inequalities are obtained by contradiction arguments.

Letting $\xi \to \infty$ in (4.17), (4.15), (4.18), and using (3.4) and recalling (3.11), shows that

$$\Delta(0) = 1-Y(0) > 0, \qquad \Delta(\xi) \to -Y(\infty) < 0 \quad \text{as} \quad \xi \to \infty. \tag{4.22}$$

Thus there exists a finite first zero $\xi^* > 0$ of $\Delta(\xi)$ such that

$$\Delta(\xi) > 0, \qquad 0 \leq \xi \leq \xi^*; \qquad \Delta(\xi^*) = 0, \tag{4.23}$$

which satisfies the requirement (4.5), and shows that the plastic wave is annulled in a finite time. Validity in the domain LOM is already confirmed.

Within the domain $MOAN$, $Y(\xi)-D(\xi,\eta) = 0$ on OA by construction, and is non-negative on OM since

$$\frac{d}{d\xi}\{Y(\xi)-D(\xi,\xi+)\} = Y'(\xi)+\{E'(\xi)-E'(2\xi)\} \geq 0, \tag{4.24}$$

in view of (3.2), (3.5). In fact it is strictly positive on OM, $\xi > 0$, if $E''(0) < 0$, or $Y'(0) > 0$. On AN, $\eta = [(1-c)/c]\xi^* + \xi$,

$$\theta(\xi,\eta) = \theta(\xi^*,\eta^*)-\frac{1}{2}E\left(\frac{1+c}{c}\xi^*\right)+\frac{1}{2}E\left(2\xi+\frac{1+c}{c}\xi^*\right), \tag{4.25}$$

which increases as ξ increases from ξ^*, approaching the limit

$$\theta_N = \theta(\xi^*,\eta^*)-\frac{1}{2}E\left(\frac{1+c}{c}\xi^*\right)$$

as $\xi \to \infty$. The increase, $-\frac{1}{2}E\{[(1+c)/c]\xi^*\}$, beyond $A(\xi^*,\eta^*)$ becomes smaller as the annullment distance ξ^* increases. Similarly, $Y(\xi)-D(\xi,\eta)$, zero at A, is non-decreasing along AN, and becomes strictly positive for $\xi > \xi^*$ if $Y'(\xi^*) > 0$ or $E''(\xi^*) < 0$. If $E''(\xi) \equiv 0$, an initial linear distribution, then at fixed ξ, $\theta(\xi,\eta)$ is monotonic increasing (strictly increasing if $Y'(\xi) > 0$) in η, so that $Y(\xi)-D(\xi,\eta)$ is decreasing, and hence is non-negative in $MOAN$. In general $\theta(\xi,\eta)$ is not monotonic in η, but on OA

$$\frac{\partial\theta}{\partial\eta}(\xi,\xi/c) = \frac{c}{1-c}\left\{Y'(\xi)+\left[E'(\xi)-E'\left(\frac{1+c}{c}\xi\right)\right]\right\} \geq 0, \tag{4.26}$$

and is strictly positive if $Y'(\xi) > 0$ or $E''(\xi) < 0$. Thus, provided that $\partial\theta/\partial\eta$ does not change sign more than once at fixed ξ, validity is satisfied, but confirmation rests on the numerical solution for any given $E(\xi)$, $Y(\xi)$. It is satisfied in the illustrating examples presented in the next section.

With the domain OAB, since $j'(z) > 0$,

$$\theta(\xi,\eta) > 0, \qquad \xi > 0, \tag{4.27}$$

and in particular, the stress behind the plastic discontinuity, at $(\xi, \xi+/c)$,

$$\theta_p(\xi) = Y(\xi)-Y(0)+1+E(\xi)+\frac{1-\delta}{2\delta}\{1+E(z)-2j(z)\}, \tag{4.28}$$

is tensile for $0 < \xi \leq \xi^*$. Thus the plastic jump does reach a tensile state B, Fig. 1. In general $\theta_p(\xi)$ does not increase monotonically, as shown in one of the examples. Furthermore, since $j''(z) \leq 0$, $\partial\theta/\partial\eta \leq 0$, and so the stress follows the path BW in the unloading sense as η increases, and certainly remains within the elastic range BC. In the case $Y \equiv$ constant, $G(z)$ (4.17), hence $j(z)$ (4.15), and $\theta_p(\xi)$ (4.28), are independent of Y for given δ, but the plastic jump $\Delta(\xi)$ (4.18) necessarily depends on Y by its construction. Thus the annullment distance ξ^*, and hence the range of arguments for $j(z)$, $\theta_p(\xi)$, depends on Y.

Again the stress is positive, and decreasing in η, in the domains $NABQ$ and QBC. In the latter this is trivial since $E'(\xi) > 0$, $E''(\xi) \leq 0$; and $\theta(\xi,\eta) \to 0$ as $\eta \to \infty$, fixed ξ, since $E(\xi) \to 0$ as $\xi \to \infty$. The stress is therefore positive on BQ, and it remains to show it is decreasing in $NABQ$. Here

$$\frac{\partial \theta}{\partial \eta} = \tfrac{1}{2}E'(\xi+\eta)-j'(\eta-\xi) \leq \tfrac{1}{2}E'(\eta-\xi)-j'(\eta-\xi). \tag{4.29}$$

By (4.15), (4.21),

$$G'(z) = j'(z)-\delta^2 j'(\delta z) \leq (1-\delta^2)j'(z), \tag{4.30}$$

and by (4.17), (3.5) and (3.2),

$$G'(z) = \tfrac{1}{2}(1-\delta)\left\{E'(z)+\delta E'\left(\frac{c}{1+c}z\right)+\delta Y'(\xi)\right\} \geq \tfrac{1}{2}(1-\delta^2)E'(z). \tag{4.31}$$

Thus $j'(z) \geq \frac{1}{2}E'(z)$ and by (4.29) the stress decreases in η, which completes the analytic validity conclusions.

5. ILLUSTRATIONS

Numerical solutions have been obtained for the following six cases, all with

$$E(\xi) = -e^{-\xi} \tag{5.1}$$

which satisfies the postulates $E''(\xi) \leq 0$, $E'''(\xi) \geq 0$.

$$\left.\begin{array}{lll} 1. & Y(\xi) \equiv 0{\cdot}5, & c = 0{\cdot}8; \\ 2. & Y(\xi) \equiv 0{\cdot}75, & c = 0{\cdot}8; \\ 3. & Y(\xi) = 1-0{\cdot}5e^{-\xi}, & c = 0{\cdot}8; \\ 4. & Y(\xi) = 1{\cdot}5(1-0{\cdot}5e^{-\xi}), & c = 0{\cdot}8. \end{array}\right\} \tag{5.2}$$

In these first four cases $\delta = 0{\cdot}111$ and two terms of the iterated series (4.19) determines $j(z)$ up to $z = 10$ with an error $\leq 1{\cdot}5\times 10^{-3}$; in fact $z^* < 5$. Cases 3 and 4 consider the yield stress to decrease linearly with temperature, by a factor 0·5 over the maximum temperature rise, and the surface values $Y(0)$ are those of cases 1 and 2 respectively.

Cases 5 and 6 are a repeat of 1 and 2 with a smaller value of c:

$$\left.\begin{array}{lll} 5. & Y(\xi) \equiv 0{\cdot}5, & c = 0{\cdot}714; \\ 6. & Y(\xi) \equiv 0{\cdot}75, & c = 0{\cdot}714; \end{array}\right\} \tag{5.3}$$

for which $\delta = 0{\cdot}167$ and two terms of (4.19) are sufficient for an error $<2\times 10^{-3}$ in $z \leq z^* \leq 4{\cdot}3$. The results for these cases show a slightly shorter annullment distance ξ^*, and a slightly lower maximum value of $\theta_p(\xi)$, $\xi \leq \xi^*$, and of the limit stress θ_N as $\xi \to \infty$, than the corresponding values in cases 1 and 2 respectively; all approximately 90 per cent. Since further decrease of c implies $G > 0{\cdot}75K$, and is uncommon, cases 1–4 with $c = 0{\cdot}8$ are presented as typical results.

Figure 3 shows the function $j\{[(1+c)/c]\xi\}$ and the stress $\theta_p(\xi)$ for cases 1 and 2, with annullment distances respectively

$$\xi_1^* = 1{\cdot}98, \qquad \xi_2^* = 0{\cdot}96; \tag{5.4}$$

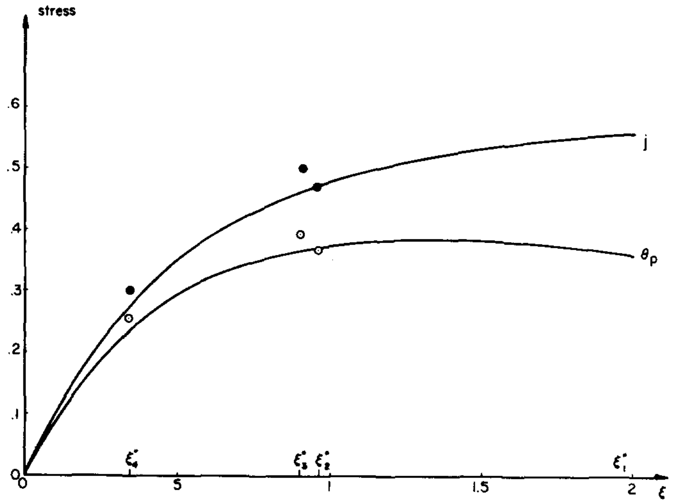

FIG. 3. Wave function $j[(1+c)\xi/c]$, and stress behind plastic discontinuity $\theta_p(\xi)$, case 1, and end points for cases 2, 3, 4.

the respective curves being common to the minimum annullment distance ξ_2^*, marked by the points. The corresponding curves for cases 3 and 4, where

$$\xi_3^* = 0{\cdot}90, \qquad \xi_4^* = 0{\cdot}34, \tag{5.5}$$

are not significantly different within their range of definition, and the end points only are marked. It is clear from the $Y(\xi)$ dependence in $G(z)$ (4.17), and the iterated solution (4.19), that for small δ a varying $Y(\xi)$ has little influence on $j(z)$.

In case 1 the stress behind the plastic discontinuity, $\theta_p(\xi)$, has a maximum 0·38 at $\xi = 1{\cdot}34 < \xi^*$, which is greater than the limit stress in the outgoing elastic wave, $\theta_N = 0{\cdot}37$. In cases 2–4 the plastic maximum is $\theta_p(\xi^*)$, which is less than the elastic maximum θ_N, as follows

$$\begin{aligned} &2.\ \theta_p(\xi_2^*) = 0{\cdot}37, \qquad \theta_N = 0{\cdot}43;\\ &3.\ \theta_p(\xi_3^*) = 0{\cdot}39, \qquad \theta_N = 0{\cdot}46;\\ &4.\ \theta_p(\xi_4^*) = 0{\cdot}26, \qquad \theta_N = 0{\cdot}48. \end{aligned} \tag{5.6}$$

For decreasing annullment distance the subsequent elastic tensile stress increase is more significant, as indicated after (4.25), and in fact the limit θ_N becomes closer to the purely elastic solution limit 0·5. However, in case 1 which takes the limit value of Y appropriate to the present wave pattern, there is a decrease in both the maximum tensile stress and the tensile stress in the outgoing elastic wave, by a factor 0·76.

Another aspect is the pulse shape of the outgoing wave. Figure 4 compares the pulse at $\xi = 2$ in cases 1–4 and the elastic solution. Here the plastic discontinuity is annulled in all cases. Apart from the different maximum stresses there is a varying spread in the tension build-up, which is least sharp in case 1. The spread is less at shorter distances ξ where the elastic and plastic tensile jumps are not separated as far in time. In each computation the stress was found to remain below yield in the elastic domain $MOAN$, Fig. 2, as

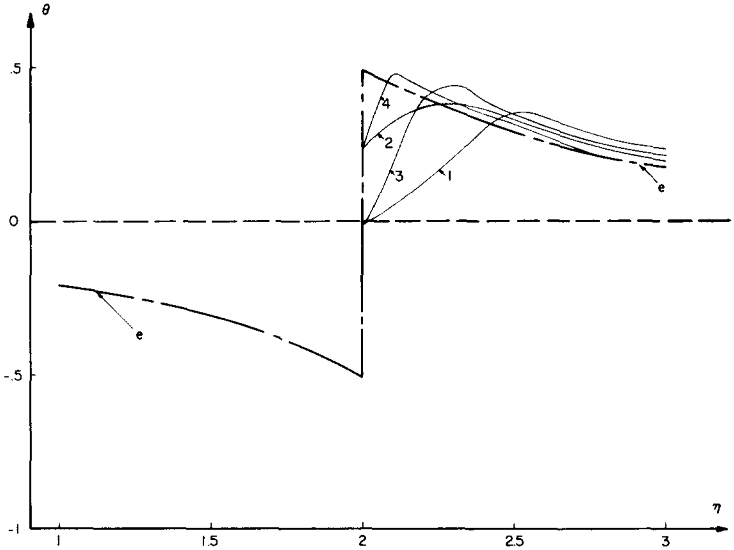

FIG. 4. Stress variation with time at $\xi = 2$: elastic (e) and four plastic cases.

required. The elastic tensile discontinuity has the constant magnitude $Y(0)$, which should be readily observable once the above spread of the remaining jump has taken place.

Finally, we may note that the preceding wave pattern will be valid for $Y(0) < \frac{1}{2}$ in some domain of the ξ–η plane not influenced by signals from the onset of reverse yield. This domain decreases as the breakdown distance ξ_R defined by (3.9) decreases, and hence as Y decreases for constant Y. In the limit $Y = 0$, $\xi_R = 0$; while $\xi_R \to \infty$ as $Y \to 0{\cdot}5$. Since $\theta_p(\xi)$ is independent of Y (constant) the maximum value obtained at $\xi = 1{\cdot}34$ in case 1 will be reached for lower Y if $[2c/(1+c)]\xi_R, \xi^*, > 1{\cdot}34$. That is, the maximum plastic tension can be further reduced only by the onset of reverse yield at sufficiently short distances ξ_R. It is anticipated though that reverse yield would attenuate the tensile stresses in the final outgoing elastic wave.

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Абстракт—Абсорпция и радиация через тонкий слой, примыкающий к облученной и свободной от напряжений поверхности, вызывает быстрый нагрев и образование волн напряжения. Исследуется рост температуры, достаточный для образования пластического течения. Приводится решение суммарных упруго-пластических волн. Дается сравнение с исключительно упругим решением. В качестве особого случая, иллюстрируется редуцированная амплитуда и распространение импульса, касающегося излученной волны растяжения.